TRINITY COLLEGE

Semester Two Examination, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words SOLUTIONS

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	100	65
			Total	153	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **(53 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

(a) If $\tan A = \frac{1}{2}$, determine the exact value of $\tan 2A$. (2 marks)

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$= \frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} \checkmark$$

$$= \frac{1}{\frac{3}{4}}$$

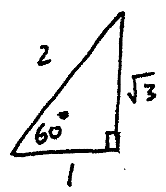
$$= \frac{4}{3} \checkmark$$

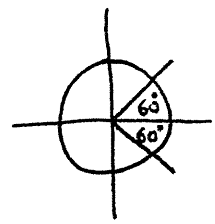
(b) Solve $\cos\big(2(x-10°)\big) = 0.5,\ 0° \le x \le 180°$. (3 marks)

$$\Rightarrow 2(x-10°) = 60° \text{ or } 300° \checkmark \;;\; -20° \le 2(x-10°) \le 340°$$

$$\Rightarrow x - 10° = 30° \text{ or } 150°$$

$$\therefore x = 40° \text{ or } 160° \quad \checkmark \checkmark$$





Question 2 **(6 marks)**

If $A=\begin{bmatrix}2\\1\\3\end{bmatrix}$, $B=\begin{bmatrix}4 & -1 & 2\end{bmatrix}$, $C=\begin{bmatrix}3 & 3 & -2\\5 & 4 & -3\end{bmatrix}$ and $D=\begin{bmatrix}1 & 2\\4 & 0\\7 & 3\end{bmatrix}$, state whether the following are true or false. If false, clearly explain your reasoning.

(a) $A+D=\begin{bmatrix}2 & 1 & 2\\1 & 4 & 0\\3 & 7 & 3\end{bmatrix}$. (1 mark)

False, can not add matrices of different dimension.

✓

(b) $AB=\begin{bmatrix}8 & -2 & 4\\4 & -1 & 2\\12 & -3 & 6\end{bmatrix}$. (1 mark)

True ✓

(c) $a_{12}+b_{21}+c_{11}+d_{22}=3$. (1 mark)

False, a_{12} and b_{21} do not exist.

✓

(d) $BA=13$. (1 mark)

False, $BA=[13]$

i.e. the product of two matrices is a matrix. ✓

i.e. $[13]\neq 13$

this is related to the concepts of: scalar Multiplication vs 'scalar matrix' Mult.

(e) $CD=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. (1 mark)

True ✓

(f) $C^{-1}=D$. (1 mark)

False, Matrix C must be square to have an inverse. ✓

Question 3 **(7 marks)**

Two vectors are given by $\mathbf{a} = 9\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$. Determine

(a) a vector parallel to $\mathbf{a} - \mathbf{b}$ of magnitude 25. (3 marks)

$$\underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ 8 \end{pmatrix} \checkmark$$

$$|\underset{\sim}{a} - \underset{\sim}{b}| = \sqrt{6^2 + 8^2}$$
$$= \sqrt{100}$$
$$= 10 \checkmark$$

$\therefore$ Required vector is

$$\frac{25}{10} \times \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$
$$= \begin{pmatrix} 15 \\ 20 \end{pmatrix}$$
$$= \underline{\underline{15\underset{\sim}{i} + 20\underset{\sim}{j}}} \checkmark$$

or $-15\underset{\sim}{i} - 20\underset{\sim}{j}$

(b) $\mathbf{a}$ in terms of $\mathbf{d}$ and $\mathbf{e}$, where $\mathbf{d} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{e} = 5\mathbf{i} - 2\mathbf{j}$. (4 marks)

$$\begin{pmatrix} 9 \\ 4 \end{pmatrix} = x\begin{pmatrix} 3 \\ -5 \end{pmatrix} + y\begin{pmatrix} 5 \\ -2 \end{pmatrix} \checkmark$$

$\Rightarrow$ $3x + 5y = 9$ and $-5x - 2y = 4$ $\checkmark$

By elimination:

$$\begin{aligned} 15x + 25y &= 45 \\ -15x - 6y &= 12 \\ \hline 19y &= 57 \end{aligned}$$

$\therefore y = 3$; $x = -2$ $\checkmark$

$$\therefore \underline{\underline{\underset{\sim}{a} = -2\underset{\sim}{d} + 3\underset{\sim}{e}}} \checkmark$$

Question 4 **(9 marks)**

(a) Evaluate $\frac{3!7!}{9!}$. (2 marks)

$= \frac{6 \times \cancel{7!}}{9 \times 8 \times \cancel{7!}}$ ✓

$= \frac{1}{12}$ ✓

(b) Determine the number of different permutations of the letters in the word NEEDLED. (2 marks)

$\frac{7!}{3!\,2!}$ ✓

$= \frac{7 \times 6 \times 5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{3!}\,\cancel{2!}}$

$= 420$ ✓

(c) A password is formed using all seven of the characters \$, %, @, Y, Z, 8 and 9 just once. Determine the number of different passwords that are possible in which all the symbols are adjacent, all the letters are adjacent and all the digits are adjacent. (3 marks)

together

$3!\ 2!\ 2!\ 3!$

Symb. Letters Digit order ✓ ✓

$= 6 \times 2 \times 2 \times 6$

$= 144$ passwords ✓

(d) Determine the least number of randomly chosen integers between 10 and 99 required to be certain that the difference of the digits in at least two of the integers is the same. (*For example, the difference of the digits in the integer* 49 *is* $9-4=5$). (2 marks)

there are 10 possible differences (pigeonholes)
(ie. a difference of 0 up to a difference of 9) ✓

∴ By pigeonhole principle we will need at least 11 numbers to be chosen. ✓

Question 5 **(5 marks)**

A proposition states that for any integer n, if $n^2 - 4n - 3$ is even, then n is odd.

(a) Write the contrapositive of this proposition. (1 mark)

If n is not odd (even), then n^2-4n-3 is not even (odd).

✓

(b) Use the contrapositive statement to prove the proposition is true. (4 marks)

If n is not odd it is even

$\therefore \quad n = 2k, \quad k \in \mathbb{Z}$ ✓

now $n^2-4n-3 = (2k)^2 - 4(2k) - 3$

$= 4k^2 - 8k - 3$

$= 4k^2 - 8k - 2 - 1$

$= 2(2k^2 - 4k - 1) - 1$ ✓

$= 2l - 1, \quad l \in \mathbb{Z}$.

which is odd and

$\therefore$ not even Q.E.D. ✓

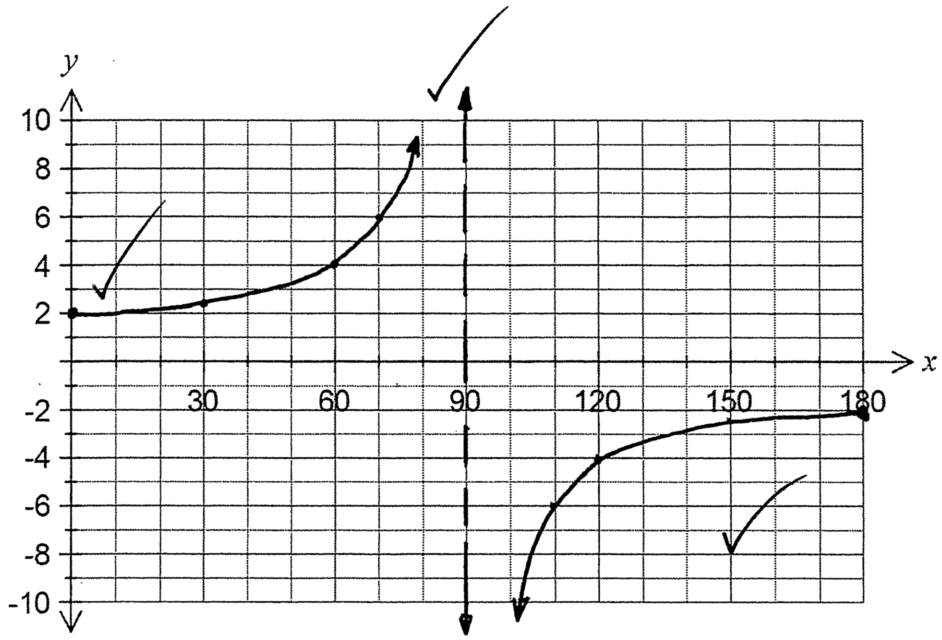
i.e. Given the contrapositive is true the original proposition must be true

Q.E.D.

✓

Question 6 **(7 marks)**

(a) Sketch the graph of $y = 2\,\mathrm{cosec}(x + 90)$ for $0° \leq x \leq 180°$. (3 marks)



(b) Prove the identity $\cot A + \tan A = \sec A\,\mathrm{cosec}\,A$. (4 marks)

Proof: Take L.H.S. $= \cot A + \tan A$

$$= \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}$$

$$= \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}$$

$$= \frac{1}{\cos A \sin A}$$

$$= \frac{1}{\cos A} \cdot \frac{1}{\sin A}$$

$$= \sec A \,.\, \mathrm{cosec}\, A$$

$= \text{RHS}$

QED.

Question 7 **(7 marks)**

(a) Matrix A represents a rotation of 180° about the origin. Determine

(i) matrix A. (1 mark)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \checkmark$$

(ii) the exact coordinates of the point (-2, 3) after transformation by matrix A. (1 mark)

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

ie. $\underline{\underline{A'(2,-3)}}$ ✓

(iii) the determinant of matrix A. (1 mark)

$$|A| = (-1)(-1) - (0)(0)$$
$$= \underline{\underline{1}} \checkmark$$

(b) Matrix $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Describe the transformation represented by B and calculate its determinant. (2 marks)

B is a reflection in the $\underline{\underline{\text{y-axis}}}$ ✓

$$|B| = \underline{\underline{-1}} \checkmark$$

(c) Use an example to show that two non-singular square matrices C and D exist such that the determinant of their sum is equal to the sum of their determinants. (2 marks)

Let $C = A$ and $D = B$ from above for example

Now $C + D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

Now $|C+D| = 0 = |C| + |D|$ ✓

$$= 1 + (-1) \quad \text{QED.}$$

Question 8 **(7 marks)**

The complex number $z = \frac{1+i}{1-i} - \frac{4+3i}{a-i}$, where a is a real constant.

(a) Show that $\text{Re}(z) = \frac{3-4a}{a^2+1}$ and that $\text{Im}(z) = \frac{a^2-3a-3}{a^2+1}$. (4 marks)

$$z = \frac{1+i}{1-i} \times \frac{1+i}{1+i} - \frac{4+3i}{a-i} \times \frac{a+i}{a+i} \quad \checkmark$$

$$= \frac{1+2i+i^2}{2} - \frac{4a+4i+3ai+3i^2}{a^2-i^2}$$

$$= \frac{1+2i-1}{2} - \frac{4a-3+(3a+4)i}{a^2+1}$$

$$= i \checkmark - \frac{4a-3}{a^2+1} - \frac{3a+4}{a^2+1} i$$

$$= \frac{3-4a}{a^2+1} + \frac{a^2+1-(3a+4)}{a^2+1} i \quad \checkmark$$

$$= \frac{3-4a}{a^2+1} + \frac{a^2-3a-3}{a^2+1} i$$

$$= \text{Re}(z) + \text{Im}(z)\, i \quad \checkmark \text{ QED.}$$

(b) Determine the value(s) of a when $\text{Im}(z) + \text{Re}(z) = 0$. (3 marks)

$$\frac{3-4a}{a^2+1} + \frac{a^2-3a-3}{a^2+1} = 0 \quad \checkmark$$

$$\Rightarrow 3-4a + a^2-3a-3 = 0$$

$$\Rightarrow a^2-7a = 0 \quad \checkmark$$

$$\Rightarrow a(a-7) = 0$$

$$\therefore \underline{a=0} \text{ or } \underline{a=7} \quad \checkmark$$

End of questions.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(6 marks)**

The work done, in joules, by a force $\mathbf{F}$ Newtons in changing the displacement of an object $\mathbf{s}$ metres is given by the scalar product of $\mathbf{F}$ and $\mathbf{s}$.

(a) Determine the work done by a force of 200 N that moves an object 2.7 m, given that the force acts at an angle of 17° to the direction of movement. (1 mark)

$$\underset{\sim}{F} \bullet \underset{\sim}{s} = |\underset{\sim}{F}||\underset{\sim}{s}|\cos\theta$$
$$= 200 \times 2.7 \times \cos 17^\circ$$
$$= \underline{\underline{516.4}} \text{ J} \quad \checkmark$$

(b) When an object is moved $0.8\mathbf{i} - 0.6\mathbf{j}$ m by a force of 130 N, the work done is 126 J.

(i) Show that one possible force is $120\mathbf{i} - 50\mathbf{j}$ N. (2 marks)

$$\underset{\sim}{F} \bullet \underset{\sim}{s} = F_1 s_1 + F_2 s_2$$

$$\begin{bmatrix} 120 \\ -50 \end{bmatrix} \bullet \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} = 96 + 30$$
$$= \underline{\underline{126 \text{ J}}} \quad \checkmark$$

where $|\underset{\sim}{F}| = \sqrt{(120)^2 + (-50)^2}$ ✓
$= \underline{\underline{130 \text{ N}}}$

(ii) Another possible force is $x\mathbf{i} + y\mathbf{j}$ N. Determine the values of x and y. (3 marks)

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} = 126$$

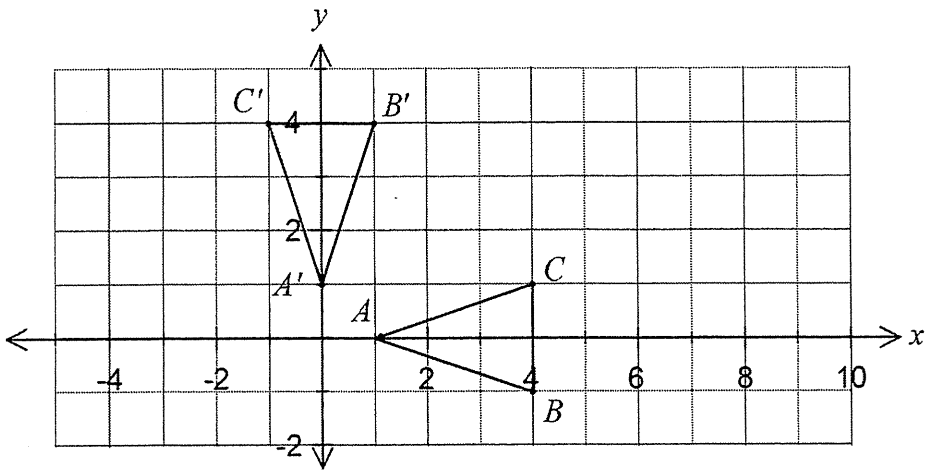
$\Rightarrow 0.8x - 0.6y = 126$ ✓ and $x^2 + y^2 = 130^2$ ✓

Solving simultaneously gives

$x = 120,\ y = -50$ given in part (i) or $\underline{\underline{x = 84.6,\ y = -101.2}}$ ✓

Question 10 **(7 marks)**

On the axes below, triangle ABC is transformed to $A'B'C'$ by a linear transformation.



(a) State the appropriate transformation matrix. (1 mark)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \checkmark$$

(b) Following a second transformation, $A'(0, 1)$ and $B'(1, 4)$ are transformed to $A''(0, 2)$ and $B''(3, 8)$.

(i) Determine the matrix for this second transformation. (2 marks)

$$T \begin{array}{c} \begin{matrix} A' & B' \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} A'' & B'' \end{matrix} \\ \begin{bmatrix} 0 & 3 \\ 2 & 8 \end{bmatrix} \end{array} \checkmark$$

$$\Rightarrow T = \begin{bmatrix} 0 & 3 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \checkmark$$

(ii) Calculate the area of triangle $A''B''C''$. (2 marks)

$$A(\triangle A''B''C'') = |\det(T)| \times A(\triangle A'B'C') \checkmark$$

$$= 6 \times \tfrac{1}{2}(2)(3)$$

$$= \underline{\underline{18 \text{ units}^2}} \checkmark$$

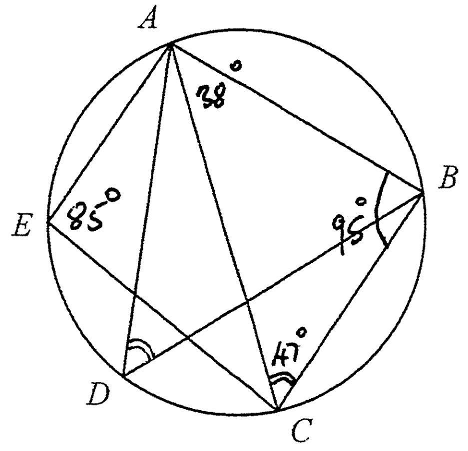
(c) Determine the transformation matrix that will transform triangle $A''B''C''$ back to ABC. (2 marks)

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \checkmark = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix} \checkmark$$

Question 11 **(8 marks)**

(a) In the diagram below $\angle AEC = 85°$ and $\angle BAC = 38°$. Determine the size of $\angle ADB$.

(3 marks)



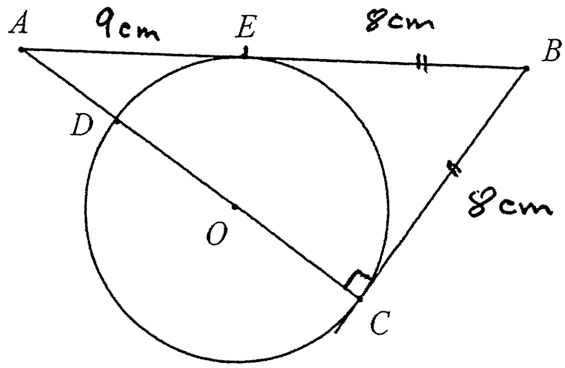
$\angle ABC = 180 - 85$
$= 95°$ ✓ cyclic quad.

$\angle ACB = 180 - 95 - 38$
$= 47°$ ✓ angle sum tri.

$\therefore \angle ADB = \angle ACB$ standing on same arc.
$= \underline{\underline{47°}}$ ✓

(b) In the diagram shown below, not drawn to scale, a circle with centre O has tangents at E and C that meet at B. If the length of BC is 8 cm and the length of AE is 9 cm, determine the length of DC.

(5 marks)



$BE = BC = 8\text{cm}$ ✓

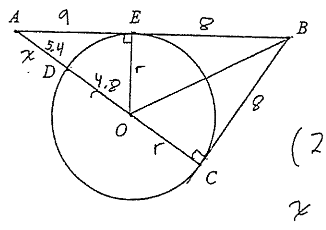
$AB = AE + EB$
$= 9 + 8$
$= 17\text{cm}$ ✓

$AC = \sqrt{(AB)^2 - (BC)^2}$
$= \sqrt{17^2 - 8^2}$
$= 15\text{cm}$ ✓

$(AE)^2 = AD \times AC$
$\Rightarrow AD = \dfrac{9^2}{15}\ \text{cm}$
$= 5.4\ \text{cm}$ ✓

$\therefore DC = AC - AD$
$= 15 - 5.4$
$= \underline{\underline{9.6\text{cm}}}$ ✓

Alternative Sol^n (Samuel Carbone)



$9^2 = x(x + 2r)$

$9^2 + r^2 = (x + r)^2$

$(2r + x)^2 + 8^2 = 17^2$

$x = 5.4,\ y = 4.8$

$\therefore DC = 2 \times 4.8$
$DC = \underline{9.6\text{cm}}$ ✓

Question 12 **(9 marks)**

Let $A = \begin{bmatrix} -6 & 4 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(a) Given that $A^{-1} = kB$, determine the value of k. (2 marks)

$\Rightarrow A^{-1}B^{-1} = kI$

$\Rightarrow kI = (BA)^{-1}$ ✓

$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$\therefore \underline{\underline{k = \frac{1}{2}}}$ ✓

Alternative method:

$A^{-1} = \begin{bmatrix} 3/2 & 2 \\ \frac{5}{2} & 3 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= \frac{1}{2}B$

(b) The equations $4y = 6x + 4$ and $5x = 3y$ can be expressed as a matrix equation in the form $AX = C$.

(i) State the matrices X and C. (2 marks)

$X = \begin{bmatrix} x \\ y \end{bmatrix}$ ✓ $C = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ ✓

(ii) Write down a matrix equation to determine X in terms of B and C. (2 marks)

$AX = C$

$\Rightarrow X = A^{-1}C$ ✓

$= \underline{\underline{\frac{1}{2}BC}}$ ✓ from (a)

(c) Determine the matrix D, if $(B - D)B = 2A$. (3 marks)

$\Rightarrow (B-D)BB^{-1} = 2AB^{-1}$ ✓

$\Rightarrow (B-D)I = 2A\frac{1}{2}A$

$\Rightarrow B - D = A^2$

$\Rightarrow D = B - A^2$ ✓

$= \begin{bmatrix} -53 & 40 \\ 50 & -23 \end{bmatrix}$ ✓

From (a)

$A^{-1} = \frac{1}{2}B$

$\Rightarrow 2A^{-1} = B$

$\Rightarrow \frac{1}{2}A = B^{-1}$

Question 13 **(6 marks)**

(a) Determine the angle between the vectors $(-12,7)$ and $(3,8)$. (2 marks)

Could use:

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

OR ClassPad

angle([-12, 7], [3, 8])

$= \underline{\underline{80.3°}}$ (1d.p.) ✓ ✓

(b) Determine the value of a so that the vectors $(7,a)$ and $(10,4)$ are perpendicular. (2 marks)

$(7, a) \cdot (10, 4) = 0$ ✓

$\Rightarrow 70 + 4a = 0$

$\therefore \quad \underline{\underline{a = -\frac{35}{2}}}$ ✓

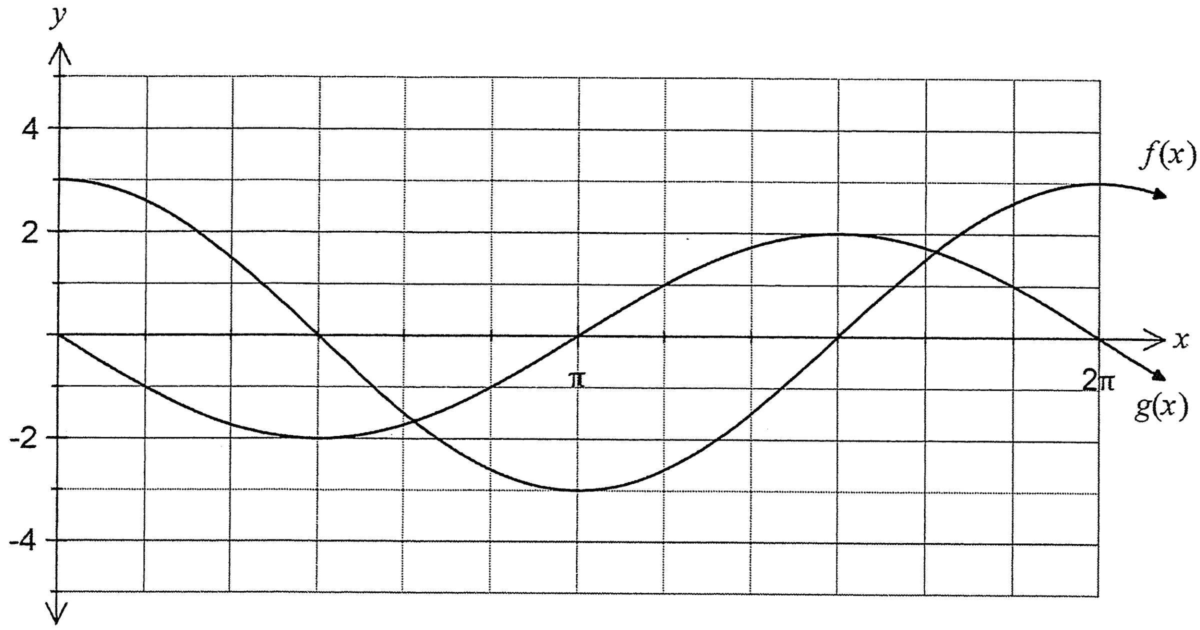
(c) Determine the exact scalar projection of $\underset{\sim}{a}$ $(3,-5)$ on $\underset{\sim}{b}$ $(-8,4)$. (2 marks)

$\text{Proj}\ \underset{\sim}{a} \text{ on } \underset{\sim}{b} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|} = \dfrac{(3,-5)\cdot(-8,4)}{|(-8,4)|}$ ✓

$= \underline{\underline{-\dfrac{11\sqrt{5}}{5}}}$ ✓

Question 14 **(9 marks)**

(a) The graphs of $y = f(x)$ and $y = g(x)$ are shown below for $0 \le x \le 2\pi$.



(i) If $f(x) = a\cos x$ and $g(x) = b\sin x$, state the values of a and b. (1 mark)

$a = 3$ $\quad$ $b = -2$

(ii) If $h(x) = f(x) + g(x)$ express $h(x)$ in the form $R\cos(x + \alpha)$. (3 marks)

$$R = \sqrt{(3)^2 + (-2)^2}$$
$$= \sqrt{13}$$

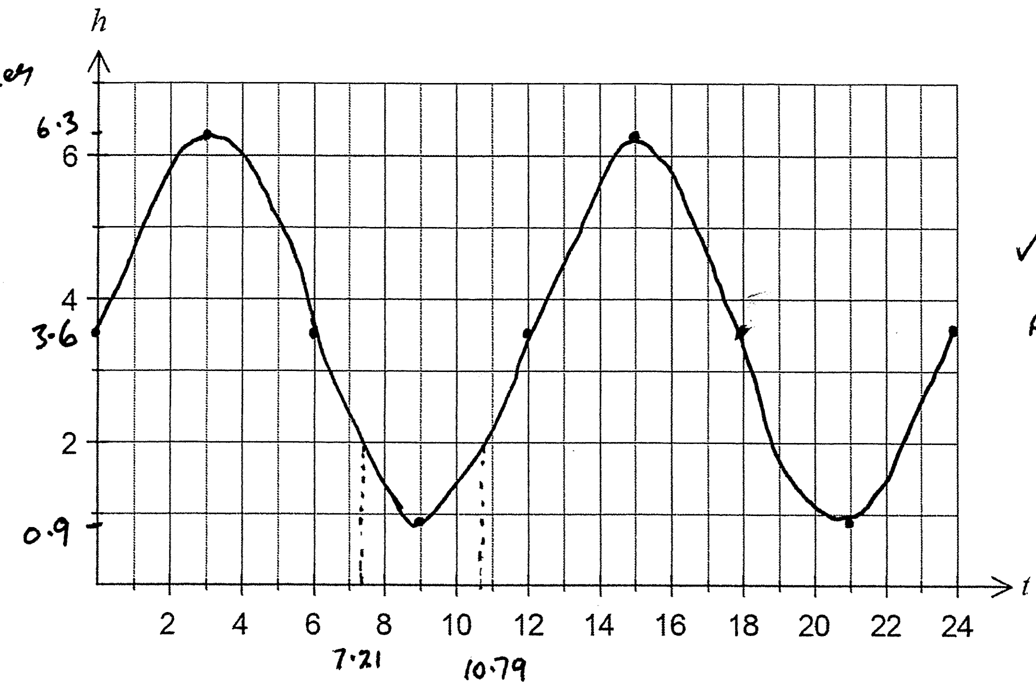
$$\alpha = \tan^{-1}\left(\frac{2}{3}\right)$$
$$= 0.588 \quad (3\text{d.p})$$

$$\therefore h(x) = \sqrt{13}\cos(x + 0.588)$$

(b) The clearance, h metres, under a bridge spanning a river estuary varies with the time since midnight, t hours, and is given by $h = 3.6 + 2.7\sin\left(\frac{\pi t}{6}\right)$.

(i) Sketch the graph of the clearance against time on the axes below. (3 marks)

Label values on axes.



show values on axes.

✓✓

Accuracy (scale)

✓

(ii) Determine the percentage of any 24-hour period during which the clearance under the bridge is no more than two metres. (2 marks)

$h \leq 2$

$\Rightarrow \quad 7.21 \leq t \leq 10.79 \quad$ (2d.p.)

✓

$\therefore \quad \dfrac{10.79 - 7.21}{12} = 0.298 \quad$ (3d.p)

$= \underline{\underline{29.8\%}} \quad$ (1d.p. of %)

✓

Question 15 **(8 marks)**

(a) A committee of eight people is to be selected from 10 junior, 14 adult and 11 senior nominations from the members of a club. Determine the number of ways the committee can be selected if

(i) there are no restrictions. (1 mark)

$$\binom{35}{8} = 23\,535\,820 \text{ committees.}$$

✓

(ii) there must be five adults and more seniors than juniors. (3 marks)

$$\overset{A}{\binom{14}{5}}\left(\overset{S}{\binom{11}{3}}\overset{J}{\binom{10}{0}} + \overset{S}{\binom{11}{2}}\overset{J}{\binom{10}{1}}\right) \checkmark\checkmark$$

$$= 2002\,(165 + 550)$$

$$= 1\,431\,430 \text{ committees} \checkmark$$

(b) Six books are to be selected for promotion in a newsletter from a choice of nine crime, seven fantasy and six romance novels. Determine the number of selections that include three fantasy or three romance novels. (4 marks)

three Fantasy $\overset{F}{\binom{7}{3}}\overset{\text{others}}{\binom{15}{3}} = 15925$ ✓

three Romance $\overset{R}{\binom{6}{3}}\overset{\text{others}}{\binom{16}{3}} = 11200$ ✓

Inclusion/Exclusion Principle.

three of each (Fantasy and Rom.) $\overset{F}{\binom{7}{3}}\overset{R}{\binom{6}{3}} = 700$ ✓

three of fantasy or romance $= 15925 + 11200 - 700$

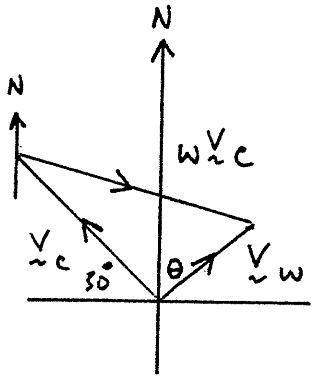
$= 26\,425$ ✓

ie. $n(F \cup R) = n(F) + n(R) - n(F \cap R)$

Question 16 **(7 marks)**

A cyclist pedals at a speed of 25 km/h along a road on a bearing of 300°. Relative to the cyclist, the wind appears to be blowing from 280° with a speed of 30 km/h.

(a) Sketch a labelled diagram to show the relationship between the velocities of the cyclist, the wind and the wind relative to the cyclist. (2 marks)



ie. $\underset{\sim}{V}_w = \underset{\sim}{V}_c + {}_w\underset{\sim}{V}_c$

✓✓

(b) Express the velocities of the cyclist and the wind relative to the cyclist in component form. (2 marks)

$$\underset{\sim}{V}_c = 25\begin{pmatrix}\cos 150° \\ \sin 150°\end{pmatrix} = \begin{pmatrix}-21.65 \\ 12.50\end{pmatrix} \quad \text{2d.p.} \quad ✓$$

$$_w\underset{\sim}{V}_c = 30\begin{pmatrix}\cos(-10) \\ \sin(-10)\end{pmatrix} = \begin{pmatrix}29.54 \\ -5.21\end{pmatrix} \quad \text{2d.p.} \quad ✓$$

(c) Determine the true speed of the wind and the bearing from which it is blowing. (3 marks)

$$\underset{\sim}{V}_w = \underset{\sim}{V}_c + {}_w\underset{\sim}{V}_c$$

$$= \begin{pmatrix}-21.65 \\ 12.50\end{pmatrix} + \begin{pmatrix}29.54 \\ -5.21\end{pmatrix}$$

$$= \begin{pmatrix}7.89 \\ 7.29\end{pmatrix} \quad ✓$$

$\therefore$ Speed $= |\underset{\sim}{V}_w| = \underline{\underline{10.7}}$ km/h ✓ Bearing $= \tan^{-1}\left(\frac{7.89}{7.29}\right)$

$\theta = \underline{\underline{047.3°}}$ (T) ✓

Question 17 **(6 marks)**

(a) Show how to express $5.\overline{25}$ as a rational number. (2 marks)

Let $x = 5.\overline{25}$

$\Rightarrow 100x = 525.\overline{25}$ ✓

$\Rightarrow 99x = 520$

$\therefore x = \frac{520}{99}$ rational number. ✓

(b) Prove by contradiction that $\sqrt[3]{4}$ is an irrational number. (4 marks)

Suppose $\sqrt[3]{4} = \frac{p}{q}$; $p, q \in \mathbb{Z}$, $q \neq 0$

ie. rational.

in lowest form (ie. p + q no common factors) ✓

$\Rightarrow 4 = \left(\frac{p}{q}\right)^3$

$\Rightarrow p^3 = 4q^3$

$\Rightarrow p^3 = 2(2q^3)$ even. ie. If p^3 is even then p must be even.

$\Rightarrow p = 2n$, $n \in \mathbb{Z}$ even. ✓

Now $4q^3 = (2n)^3$

$= 8n^3$

$\Rightarrow q^3 = 2n^3$ even ✓ " "

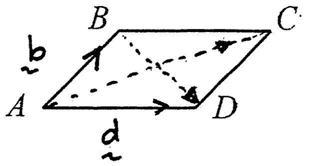
$\Rightarrow q = 2m$, $m \in \mathbb{Z}$ even

But if p and q are even, this contradicts they have no common factors.

thus $\sqrt[3]{4}$ is irrational QED. ✓

Question 18 **(9 marks)**

(a) Figure $ABCD$ is a parallelogram. Let $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Prove that the diagonals AC and BD are perpendicular only when $|\mathbf{b}| = |\mathbf{d}|$. (4 marks)



$\overrightarrow{AC} = b + d$

$\overrightarrow{BD} = d - b$ ✓

$AC \perp BD \Rightarrow \overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$\Rightarrow (b+d)\cdot(d-b) = 0$ ✓

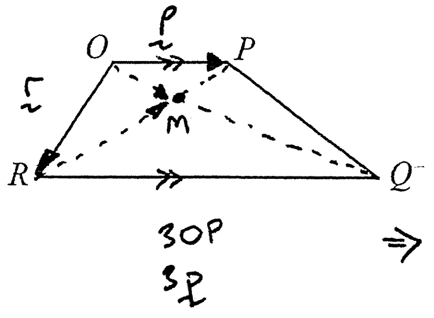
$\Rightarrow \cancel{b\cdot d} - b\cdot b + d\cdot d - \cancel{d\cdot b} = 0$ ✓

$\Rightarrow -|b|^2 + |d|^2 = 0$

$\Rightarrow |d|^2 = |b|^2$

$\therefore |d| = |b|$ QED ✓

(b) Figure $OPQR$ is a trapezium, with OP parallel to RQ and $RQ = 3OP$. If M is the point of intersection of OQ and PR, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$, $\overrightarrow{OM} = \lambda\overrightarrow{OQ}$ and $\overrightarrow{RM} = \mu\overrightarrow{RP}$ show that $\overrightarrow{OM} = \frac{1}{4}\mathbf{r} + \frac{3}{4}\mathbf{p}$. (5 marks)



$\overrightarrow{OM} = \lambda\overrightarrow{OQ}$ given

$\Rightarrow \lambda\overrightarrow{OQ} = \overrightarrow{OR} + \mu\overrightarrow{RP}$ ✓

$\Rightarrow \lambda(r + 3p) = r + \mu(p - r)$ ✓

$\Rightarrow \lambda r + 3\lambda p = (1-\mu)r + \mu p$

Equating coefficients

$\lambda = 1 - \mu$ and $3\lambda = \mu$ ✓

$\Rightarrow \lambda = 1 - 3\lambda$

$\therefore \lambda = \frac{1}{4}$ ✓

$\therefore \overrightarrow{OM} = \frac{1}{4}(r + 3p)$

$= \frac{1}{4}r + \frac{3}{4}p$ QED. ✓

Question 19 **(10 marks)**

(a) Solve $2(z-3)^2+2=0$. (2 marks)

$\Rightarrow (z-3)^2+1=0$

$\Rightarrow (z-3)^2 = -1$ ✓

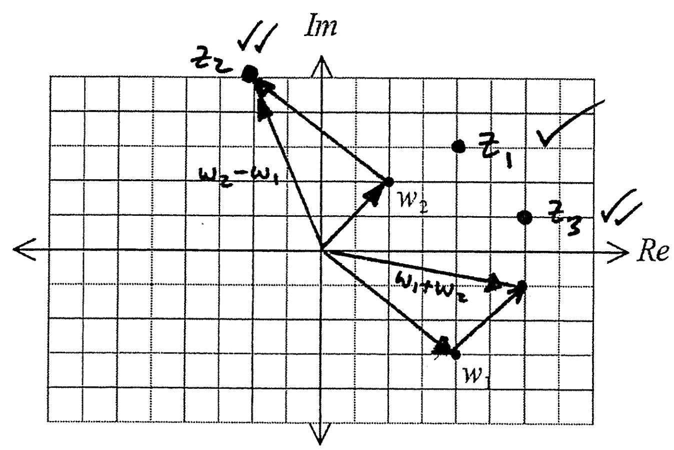
$\Rightarrow (z-3)^2 = i^2$

$\therefore\ z = 3 \pm i$ ✓

OR Using ClassPad in Cplx Mode

solve $(2(z-3)^2+2=0,\ z)$

(b) The complex numbers w_1 and w_2 are shown in the Argand plane below.



Plot and label the complex numbers given by

(i) $z_1 = \bar{w}_1$. (1 mark)

(ii) $z_2 = w_2 - w_1$. (2 marks)

(iii) $z_3 = \overline{w_1 + w_2}$. (2 marks)

(c) One solution of the quadratic equation $x^2+bx+c=0$ is $x=3-2i$. Determine the values of the real coefficients b and c. (3 marks)

$(x-(3-2i))(x-(3+2i)) = x^2+bx+c$ ✓

$\Rightarrow x^2 - x(3+2i) - x(3-2i) + 3^2 + 2^2 = x^2+bx+c$

$\Rightarrow x^2 - 3x - \cancel{2xi} - 3x + \cancel{2xi} + 13 = x^2+bx+c$

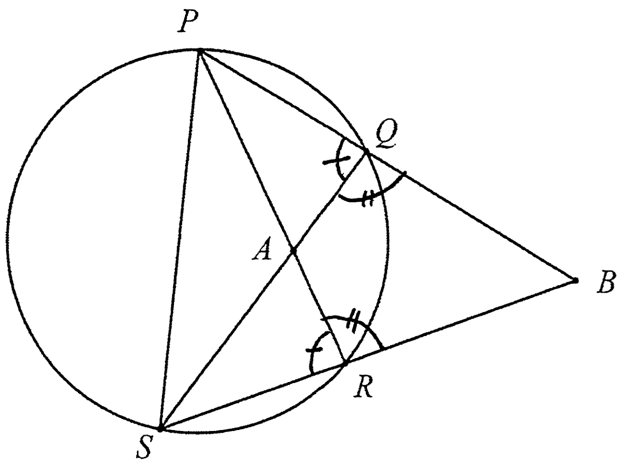
$\Rightarrow x^2 - 6x + 13 = x^2+bx+c$

$\therefore\ b=-6,\quad c=13$

✓ ✓

Question 20 **(8 marks)**

The points P, Q, R and S lie on a circle of radius r. PR and QS meet at A. PQ and SR are produced to meet at B, and $AQBR$ is a cyclic quadrilateral.



(a) Prove that BS is perpendicular to PR. (6 marks)

$\angle PQS = 180° - \angle BQS$
$\angle PRS = 180° - \angle PRB$
} straight angles supp. ✓

also $\angle PQS = \angle PRS$ on same arc ✓

$\Rightarrow 180° - \angle BQS = 180° - \angle PRB$

$\Rightarrow \angle BQS = \angle PRB$ ✓

also $\angle BQS + \angle PRB = 180°$ (Opp. angles in Cyclic Quad) additional given ✓

$\Rightarrow 2\angle PRB = 180°$

$\therefore \angle PRB = 90°$ ✓

ie. $BS \perp PR$ QED. ✓

(b) Prove that the length of PS is $2r$. (2 marks)

From (a) $\angle PRB = 90° = \angle PRS$ ✓

$\Rightarrow$ PS is a diameter ✓ (angle in semi circle)

$\therefore PS = 2r$ QED

Question 21 **(7 marks)**

Let $P(n) = 10^n + 18n - 1$.

(a) If $P(1) = 9a$ and $P(2) = 9b$, evaluate a and b. (2 marks)

$P(1) = 27 = 9a$ $\quad$ $P(2) = 135 = 9b$

$\therefore \underline{\underline{a = 3}}$ ✓ $\quad$ $\therefore \underline{\underline{b = 15}}$ ✓

(b) Prove by induction that $P(n)$ is always a multiple of nine when n is a positive integer. (5 marks)

$P(1) = 9(3)$ mult. of 9. ✓

$P(k) = 10^k + 18k - 1$
$= 9m \quad m \in \mathbb{Z}$ assumed. ✓

$P(k+1) = 10^{k+1} + 18(k+1) - 1$ ✓
$= 10 . 10^k + 18k + 18 - 1$
$= 10^k + 18k - 1 + 18 + 9 . 10^k$ ✓
$= 9m + 9(2 + 10^k)$
$= 9(m + 2 + 10^k)$

mult. of 9. ✓

ie. $P(1)$ true
$P(k)$ assumed
$P(k+1)$ true

$\therefore P(n)$ true $\forall\, n \in \mathbb{Z}^+$ QED.

End of questions.